Homework #1: Tensor manipulations
Tutorial: **10 October 2017** at 14:15

You can discuss these problems with other students, but everybody must do and present their own answers. You can use computers etc. to perform the algebraic operations, but you must show the intermediate steps (and "computer said so" is never a valid answer). You are of course free to use material from the Internet, but again, you must present the intermediate steps and you must also be able to explain why the steps are valid and why you chose them. You can mark an answer even if it is not complete or correct, as long as you have made significant progress towards solving it. Note, however, that the lecturer does the final decision on whether your solution is complete (or correct) enough for a mark.

Problem 1 (Matricization). Let $\boldsymbol{\mathcal{T}} \in \mathbb{R}^{5\times3\times3}$ be a tensor whose frontal slices are

$$\boldsymbol{T}_1 = \begin{pmatrix} 1 & 11 & 29 \\ 2 & 13 & 31 \\ 3 & 17 & 37 \\ 5 & 19 & 41 \\ 7 & 23 & 43 \end{pmatrix}, \quad \boldsymbol{T}_2 = \begin{pmatrix} 47 & 71 & 97 \\ 53 & 73 & 101 \\ 59 & 79 & 103 \\ 61 & 83 & 107 \\ 67 & 89 & 109 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{T}_3 = \begin{pmatrix} 113 & 149 & 173 \\ 127 & 151 & 179 \\ 131 & 157 & 181 \\ 137 & 163 & 191 \\ 139 & 167 & 193 \end{pmatrix}. \tag{1.1}$$

Write open

a) the matrix $\boldsymbol{T}_{(1)}$

b) the first five columns and the total dimensions of $\boldsymbol{T}_{(2)}$

c) the first five columns and the total dimensions of $\boldsymbol{T}_{(3)}$

Problem 2 (Tensor-vector product). Let $\boldsymbol{\mathcal{T}} \in \mathbb{R}^{5\times3\times3}$ be as in (1.1) and let

$$\boldsymbol{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \quad \text{and} \quad \boldsymbol{w} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

Calculate

a) $\boldsymbol{\mathcal{T}} \mathbin{\bar{\times}_1} \boldsymbol{v}$

b) $\boldsymbol{\mathcal{T}} \mathbin{\bar{\times}_3} \boldsymbol{w}$.

Problem 3 (Tensor-matrix product). Let $\boldsymbol{\mathcal{T}} \in \mathbb{R}^{5\times3\times3}$ be as in (1.1) and let

$$\boldsymbol{M} = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix}.$$

Calculate $\boldsymbol{\mathcal{T}} \times_1 \boldsymbol{M}$.

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Problem 4 (Tensors as multilinear maps). Matrices are often presented as linear maps between two vector spaces, $\boldsymbol{M}\colon \mathbb{R}^m \to \mathbb{R}^n$ with $\boldsymbol{M}(\boldsymbol{v}) = \boldsymbol{M}\boldsymbol{v}$. In some literature (especially physics), tensors are considered as *multi-linear* maps from product vector spaces to real numbers. For example, if $\boldsymbol{\mathcal{T}} \in \mathbb{R}^{I \times J \times K}$, we interpret $\boldsymbol{\mathcal{T}}\colon \mathbb{R}^I \times \mathbb{R}^J \times \mathbb{R}^K \to \mathbb{R}$. The multi-linear map is defined as

$$\boldsymbol{\mathcal{T}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{\mathcal{T}} \,\bar{\times}_1\, \boldsymbol{u} \,\bar{\times}_1\, \boldsymbol{v} \,\bar{\times}_1\, \boldsymbol{u}\ .$$

Let $\boldsymbol{\mathcal{T}}$ be as in (1.1) and let

$$\boldsymbol{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}\ , \quad \boldsymbol{v} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}\ , \quad \text{and} \quad \boldsymbol{w} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}\ .$$

Calculate $\boldsymbol{\mathcal{T}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$.